cular surfaces so narrow vertically, in proportion to their breadth, as they are in the cervical vertebræ of the *Pterosauria*: in the dorsal series the cup and ball present more ordinary Saurian proportions.

Besides these principal and more general characters, those also which distinguish the vertebræ of the several regions of the spine, together with the specialities of the atlas and axis, and of other individual vertebræ, are pointed out and described.

The Paper is illustrated by numerous figures, which (excepting two from the *Aptenodytes*) belong to the Pterodactyle.

March 31, 1859.

Sir BENJAMIN C. BRODIE, Bart., President, in the Chair.

The following communications were read:—

I. "The Higher Theory of Elliptic Integrals, treated from Jacobi's Functions as its basis." By F. W. Newman, Esq., M.A., Professor of Latin in University College, London. Communicated by the Rev. Dr. Booth. Received March 3, 1859.

(Abstract.)

The peculiarly beautiful properties of these integrals, as treated by Jacobi and (in his two supplements) by Legendre, are obtained through so very elaborate and difficult a process, that few students can afford the time to study them. Professor De Morgan, in his 'Integral Calculus,' declines to enter even the Lower Theory, on the ground that the subject requires a detailed treatise. That in some sense it is analogous to trigonometry, which no one would desire to be treated fully in the differential and integral calculus, has been recognized by several writers. Legendre, in his second supplement, sixth section, took the first steps toward treating Jacobi's functions (Λ and Θ) on a wholly independent basis, by investigating their properties from the series which they represent: but after only two pages of this sort, he aids his research by assuming their relations

to elliptic integrals as already established, which shows that he was not seeking for a new basis of argument, but only for new properties. The author of the present paper proposes (for didactic purposes) to commence the higher theory from these functions. The first division of his essay is purely algebraic and trigonometrical, not introducing the idea of elliptic integrals at all. Adopting as the definition of the functions Λ and Θ the two equations

$$\left.\begin{array}{l}\Lambda(q, x)=2q^{\frac{1}{4}}(\sin x-q^{1.2}\sin 3x+q^{2.3}\sin 5x-\&c.)\\ \Theta(q, x)=1-2q^{1\cdot 1}\cos 2x+2q^{2.2}\cos 4x-2q^{3\cdot 3}\cos 6x+\&c.\end{array}\right\},$$

it demonstrates by direct algebraic methods many properties of great generality, of which we shall here specify—

1. If $\sqrt{b}$ stands for $\dfrac{\Theta(q, 0)}{\Theta(q, \frac{1}{2}\pi)}$ and $\sqrt{c}$ for $\dfrac{\Lambda(q, \frac{1}{2}\pi)}{\Theta(q, \frac{1}{2}\pi)}$, which is shown to yield $b^2+c^2=1$; and if, further, Λ^0 Θ^0 stand for $\Lambda(q, x+\frac{1}{2}\pi)$, $\Theta(q, x+\frac{1}{2}\pi)$; we get the four equations (equivalent to two only)

$$\begin{array}{ll}\Lambda^2+b\Lambda^{02}=c\Theta^2; & \Lambda^{02}+b\Lambda^2=c\Theta^{02};\\ \Theta^2-b\Theta^{02}=c\Lambda^2; & \Theta^{02}-b\Theta^2=c\Lambda^{02};\end{array}$$

from which it directly follows, that if ω is an arc defined by the equation $\sqrt{b}\tan\omega=\dfrac{\Lambda}{\Lambda^0}$, we shall have simultaneously $\sqrt{c}\sin\omega=\dfrac{\Lambda}{\Theta}$; $\sqrt{c}\cos\omega=\sqrt{b}\dfrac{\Lambda^0}{\Theta}$; $\sqrt{(1-c^2\sin^2\omega)}=\sqrt{b}\dfrac{\Theta^0}{\Theta}$. The symbol $\Delta(c, \omega)$, $\Delta(\omega)$ or Δ represents $\sqrt{(1-c^2\sin^2\omega)}$ in this theory.

2. It is further shown that

$$\Lambda^0\frac{d\Lambda}{dx}-\Lambda\frac{d\Lambda^0}{dx}=\Lambda^2(\tfrac{1}{2}\pi).\Theta\Theta^0;$$

whence is easily obtained

$$\frac{d\omega}{dx}\propto\Delta(c, \omega).\quad \text{Also } \Theta^0\frac{d\Theta}{dx}-\Theta\frac{d\Theta^0}{dx}=\Lambda^2(\tfrac{1}{2}\pi).\Lambda\Lambda^0.$$

3. By direct multiplication of two trigonometrical series, it is found that

$$\begin{aligned}\Lambda(q, x)\Lambda^0(q, y)&=\Lambda(q^2, x-y).\Theta(q^2, x+y)\\ &\quad+\Theta(q^2, x-y).\Lambda(q^2, x+y);\\ \Theta(q, x)\Theta^0(q, y)&=\Lambda(q^2, x-y).\Lambda(q^2, x+y)\\ &\quad+\Theta(q^2, x-y).\Theta(q^2, x+y).\end{aligned}$$

From the property marked (2) we obtain the connexion of the func-

tions Λ, Θ with elliptic integrals. For, if $F(c, \omega)$, as usual, stands for

$$\int_0 \frac{d\omega}{\sqrt{(1-c^2\sin^2\omega)}},$$

it yields

$$F(c, \omega) \propto x\,;\ \text{or}\ \frac{F(c, \omega)}{F(c, \frac{1}{2}\pi)}=\frac{x}{\frac{1}{2}\pi}.$$

This introduces the second and principal part of the essay. An easy inference from (3) is, that

$$\frac{\Lambda(q^2, x+y)}{\Theta(q^2, x+y)}=\frac{\Lambda x\Lambda^0 y+\Lambda y\Lambda^0 x}{\Theta x\Theta^0 y+\Theta y\Theta^0 x};$$

and consequently that if η is related to q^2 and to $x+y$ by the same law as ω is to q and to x, while $c_{,}$ is to q^2 what c is to q, we obtain

4. $$\sqrt{c_{,}}\sin\eta=\frac{c\sin(\omega+\theta)}{\Delta(c, \omega)+\Delta(c, \theta)},$$

when

$$\frac{F(c_{,}, \eta)}{F(c_{,}, \frac{1}{2}\pi)}=\frac{F(c, \omega)+F(c, \theta)}{F(c, \frac{1}{2}\pi)}.$$

This formula has the peculiarity of comprising Euler's integrals, with the integrations of Lagrange and of Gauss; namely, if $\omega=\theta$, we get the scale of Lagrange. If $\theta=0$, the scale of Gauss is obtained. But if we introduce a new variable ζ, such that

$$F(c, \zeta)=F(c, \omega)+F(c, \theta),$$

we eliminate η by aid of the last result

$$\left(\text{which is } \sqrt{c_{,}}\sin\eta=\frac{c\sin\zeta}{\Delta(c, \zeta)+1}\right),$$

and obtain

$$\sqrt{\frac{1-\Delta\zeta}{1+\Delta\zeta}}=\frac{c\sin(\omega+\theta)}{\Delta\omega+\Delta\theta};$$

which is equivalent to Euler's integration.

The author believes this generalization to be new.

5. He proceeds (assuming now the theory of Lagrange's scale) to prove the higher theorems by much simpler processes. E being the *second* elliptic integral, he writes G for $E-\frac{E_c}{F_c}F$, and V for $\int_0 G\,dF$, and out of the integration $\frac{1}{2}\log\Delta=\frac{1}{2}V_{,}-V$ (where $V_{,}$ is to q^2 and $2x$, what V is to q and x), he deduces

$$V=\log.\frac{\Theta(q, x)}{\Theta(q, 0)}$$

by a process fundamentally that of Legendre, Second Supplement, § 196. This is the equation by which E, and indirectly the third integral Π, is linked to the functions ΛΘ.

6. We may further point out, as perhaps new, the developments of ΛΘ in the case when q is very near to 1. Let r be related to b as q to c; then $\log\frac{1}{q}.\log\frac{1}{r}=\pi^2$. If $\log\frac{1}{q}=\pi a$, and $x=\pi u$,

$$\left.\begin{array}{l}\sqrt{a}\,.\,\Theta(q,\,\pi u+\tfrac{1}{2}\pi)=r^{u^2}+r^{(1\pm u)^2}+r^{(2\pm u)^2}+r^{(3\pm u)^2}+\&\text{c}\ldots\\ \sqrt{a}\,.\,\Lambda(q,\,\pi u+\tfrac{1}{2}\pi)=r^{u^2}-r^{(1\pm u)^2}+r^{(2\pm u)^2}-r^{(3\pm u)^2}+\&\text{c}\ldots\end{array}\right\}$$

in which the double sign denotes *two* terms, which must both be included. But besides, if the symbol $\phi(r)$ stand for

$$[(1-r^2)(1-r^4)(1-r^6)\ldots\&\text{c.}]^{-1},$$

$$\left.\begin{array}{l}\sqrt{a}\,.\phi(r)\,.\,\Theta(q,\,\pi u+\tfrac{1}{2}\pi)=r^{u^2}.(1+r^{1\pm 2u})(1+r^{3\pm 2u})(1+r^{5\pm 2u})\ldots\\ \sqrt{a}\,.\phi(r)\,.\,\Lambda(q,\,\pi u+\tfrac{1}{2}\pi)=r^{u^2}.(1-r^{1\pm 2u})(1-r^{3\pm 2u})(1-r^{5\pm 2u})\ldots\end{array}\right\}.$$

From these formulæ not only all of Gudermann's developments for calculating elliptic integrals in every case are deducible, but others also, it seems, of a remarkable aspect, in the difficult case of q and c being extremely near to 1.

We produce the two which seem to be simplest. Let B be to b what C is to c, and Tan x represent $\frac{\text{Sin}\,x}{\text{Cos}\,x}$, where 2 Sin x stands for $\epsilon^x-\epsilon^{-x}$ and 2 Cos x for $\epsilon^x+\epsilon^{-x}$. Then when c is very near to 1, we compute G and thereby E from the series,

$$\begin{aligned}\text{B}\,.\,\text{G}(c,\,\omega)=\frac{\pi-2x}{\pi}&-\left(1-\text{Tan}\frac{x}{a}\right)+\left(1-\text{Tan}\frac{\pi-x}{a}\right)\\ &-\left(1-\text{Tan}\frac{\pi+x}{a}\right)+\left(1-\text{Tan}\frac{2\pi-x}{a}\right)\\ &-\left(1-\text{Tan}\frac{2\pi+x}{a}\right)+\left(1-\text{Tan}\frac{3\pi-x}{a}\right)-\&\text{c}\ldots.\end{aligned}$$

The *third* elliptic integral is in the same case deduced from a series of the form

$$\begin{aligned}&\frac{\pi-2x}{\pi}.j-\left\{j-\tan^{-1}\left(\tan j\,.\,\text{Tan}\frac{x}{a}\right)\right\}+\left\{j-\tan^{-1}\left(\tan j\,.\,\text{Tan}\frac{\pi-x}{a}\right)\right\}\\ &\quad-\left\{j-\tan^{-1}\left(\tan j\,.\,\text{Tan}\frac{\pi+x}{a}\right)\right\}+\left\{j-\tan^{-1}\left(\tan j\,.\,\text{Tan}\frac{2\pi-x}{a}\right)\right\}\\ &\quad-\&\text{c}\ldots.\end{aligned}$$

Finally, the essay developes above thirty series which rise out of this theory, nearly all of which are believed to be new. The most elegant of them may find a place here. Writing, for conciseness, C so related to c that $F(c, \frac{1}{2}\pi) = \frac{1}{2}\pi C$, and $\therefore F(c\, \omega) = Cx$, we have

$$(a).\quad \omega = x + \frac{\sin 2x}{\operatorname{Cos} \pi a} + \tfrac{1}{2}\cdot\frac{\sin 4x}{\operatorname{Cos} 2\pi a} + \tfrac{1}{3}\cdot\frac{\sin 6x}{\operatorname{Cos} 3\pi a} + \&c.$$

$$(b).\quad C\Delta(c, \omega) = 1 + \frac{2\cos 2x}{\operatorname{Cos} \pi a} + \frac{2\cos 4x}{\operatorname{Cos} 2\pi a} + \frac{2\cos 6x}{\operatorname{Cos} 3\pi a} + \&c.,$$

$$(c).\quad \tfrac{1}{8}C^2c^2 \sin 2\omega = \frac{\sin 2x}{\operatorname{Cos}\pi a} + \frac{2\sin 4x}{\operatorname{Cos} 2\pi a} + \frac{3\sin 6x}{\operatorname{Cos} 3\pi a} + \&c..$$

$$(d).\quad V(c, \omega) = \frac{1-\cos 2x}{\sin \pi a} + \tfrac{1}{2}\cdot\frac{1-\cos 4x}{\sin 2\pi a} + \tfrac{1}{3}\cdot\frac{1-\cos 6x}{\sin 3\pi a} + \&c...$$

$$(e).\quad \tfrac{1}{2}C.G(c, \omega) = \frac{\sin 2x}{\sin \pi a} + \frac{\sin 4x}{\sin 2\pi a} + \frac{\sin 6x}{\sin 3\pi a} + \&c....$$

This is virtually eq. 49 of Legendre's Second Supplement, § 7. In eq. 53 of the same, he has a development of $\sin^2 \omega$, which is given by Mr. Newman in a notation similar to eq. (c) above.

$$(f).\quad -\tfrac{1}{2}\log \Delta(c, \omega) = \frac{1-\cos 2x}{\operatorname{Sin} \pi a} + \tfrac{1}{3}\cdot\frac{1-\cos 6x}{\operatorname{Sin} 3\pi a} + \tfrac{1}{5}\cdot\frac{1-\cos 10x}{\operatorname{Sin} 5\pi a} + \&c...$$

$$(g).\quad \tfrac{1}{2}Cc \sin \omega = \frac{\sin x}{\operatorname{Sin} \frac{1}{2}\pi a} + \frac{\sin 3x}{\operatorname{Sin} \frac{3}{2}\pi a} + \frac{\sin 5x}{\operatorname{Sin} \frac{5}{2}\pi a} + \&c....$$

$$(h).\quad \tfrac{1}{2}Cc \cos \omega = \frac{\cos x}{\operatorname{Cos} \frac{1}{2}\pi a} + \frac{\cos 3x}{\operatorname{Cos} \frac{3}{2}\pi a} + \frac{\cos 5x}{\operatorname{Cos} \frac{5}{2}\pi a} + \&c....$$

$$(i).\quad \tfrac{1}{4}Cc\sqrt{\frac{1-\Delta}{1+\Delta}} = \frac{\sin x}{\operatorname{Sin} \pi a} + \frac{\sin 3x}{\operatorname{Sin} 3\pi a} + \frac{\sin 5x}{\operatorname{Sin} 5\pi a} + \&c....$$

Moreover, Jacobi's two celebrated theorems follow as a corollary from the general propositions here established.

II. "On the Comparison of Hyperbolic Arcs." By C. W. Merrifield, Esq. Communicated by the Rev. Dr. Booth. Received March 3, 1859.

(Abstract.)

If in common trigonometry we take one arc equal to the sum of two others, the cosine of the first arc is equal to the product of the cosines, diminished by the product of the sines of the other two.